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Some Inequalities Involving π and an Application to Hilbert's Inequality

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Abstract—Some inequalities involving π are proved. As an application, we give some strengthened Hilbert's inequalities.

Keywords—Hilbert inequality, Carleman inequality, Euler summation formula.

1. INTRODUCTION

Let $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$. Then Karlson's inequality is

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} n^2 a_n^2; \tag{1.1}$$

the constant π^2 cannot be made smaller, but it may be strengthened as (see [1])

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 < \pi^2 \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} \left(n - \frac{1}{2}\right)^2 a_n^2. \tag{1.2}$$

In recent years, some strengthened Hardy-Hilbert's inequalities and Carleman's inequality have been proved by introducing some weight coefficients (see [2–4]). In this paper, we prove some inequalities involving π by using the following weight coefficient:

$$\omega(n) = \sum_{m=0}^{\infty} \frac{1}{(m+n+1)} \left(\frac{n+1}{m+1}\right)^{1/2}, \tag{1.3}$$

where $n \in N_0 = NU\{0\}$ and N is the set of positive integers.

As an application, a strengthened Hilbert's inequality of the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m b_n}{(m+n+1)} < \left\{\sum_{n=0}^{\infty} \left[\pi - \frac{7}{5\left(\sqrt{n}+3\right)}\right] a_n^2 \sum_{n=0}^{\infty} \left[\pi - \frac{7}{5\left(\sqrt{n}+3\right)}\right] b_n^2\right\}^{1/2} \tag{1.4}$$

is proved.

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2. EULER'S SUMMATION FORMULA AND SOME LEMMAS

If $f(x)$ is of constant sign for $x > 0$, and, together with all its derivatives, tends monotonely to 0 as $x \to \infty$, then the Euler summation formula is

$$\sum_{m=0}^{n} f(m) = \int_0^n f(x)\,dx + \frac{1}{2}(f(n)+f(0)) + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}\left(f^{(2i-1)}(n) - f^{(2i-1)}(0)\right) + \theta \frac{B_{2k+2}}{(2k+2)!}\left(f^{(2k+1)}(n) - f^{(2k+1)}(0)\right), \tag{2.1}$$

where $0 < \theta < 1$, B_{2i} $(i \in N)$ are Bernoulli numbers (see [5]).

In particular, if $k = 0$ and $\int_0^\infty f(x)\,dx < \infty$ and $f'(x) < 0$, by (2.1), we find as $n \to \infty$,

$$\int_0^\infty f(x)\,dx + \frac{1}{2}f(0) < \sum_{m=0}^{\infty} f(m) < \int_0^\infty f(x)\,dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{2.2}$$

Let $f(x) = 1/((x+n+1)\sqrt{x+1})$, $x \in [0,\infty)$, $(n \in N)$. Then we find that $f(0) = 1/(n+1)$,

$$f'(x) = \frac{-1}{(x+n+1)^2\sqrt{x+1}} - \frac{1}{2(x+n+1)(x+1)^{3/2}}, \qquad f'(0) = \frac{-1}{(n+1)^2} - \frac{1}{2(n+1)},$$

and

$$\int_0^\infty f(x)\,dx = \int_1^\infty \frac{2dy}{n+y^2} = \frac{\pi}{\sqrt{n}} - \frac{2}{\sqrt{n}}\tan^{-1}\frac{1}{\sqrt{n}}, \qquad (n \geq 1).$$

By (2.2), for $n \in N$, we find

$$\frac{\pi}{\sqrt{n}} - \frac{2}{\sqrt{n}}\tan^{-1}\frac{1}{\sqrt{n}} + \frac{1}{2(n+1)} < \sum_{m=0}^{\infty} \frac{1}{(m+n+1)\sqrt{n+1}} < \frac{\pi}{\sqrt{n}} - \frac{2}{\sqrt{n}}\tan^{-1}\frac{1}{\sqrt{n}} + \frac{13}{24(n+1)} + \frac{1}{12(n+1)^2}. \tag{2.3}$$

Then, by (1.3) and (2.3), we have

$$\pi\left(1+\frac{1}{n}\right)^{1/2} - 2\left(1+\frac{1}{n}\right)^{1/2}\tan^{-1}\frac{1}{\sqrt{n}} + \frac{1}{2\sqrt{n}}\left(1+\frac{1}{n}\right)^{-1/2} < \omega(n) < \pi\left(1+\frac{1}{n}\right)^{1/2} - 2\left(1+\frac{1}{n}\right)^{1/2}\tan^{-1}\frac{1}{\sqrt{n}} + \frac{13}{24\sqrt{n}} \times \left(1+\frac{1}{n}\right)^{-1/2} + \frac{1}{12n^{3/2}}\left(1+\frac{1}{n}\right)^{-3/2}, \qquad (n \in N). \tag{2.4}$$

It is obvious that

$$\frac{1}{12n^{3/2}}\left(1+\frac{1}{n}\right)^{-3/2} < \frac{1}{12n^{3/2}}, \qquad (n \in N).$$

By Taylor's formula, we find

$$1 + \frac{1}{2n} - \frac{1}{8n^2} < \left(1+\frac{1}{n}\right)^{1/2} < 1 + \frac{1}{2n}, \qquad (n \geq 3),$$

$$\frac{1}{\sqrt{n}} + \frac{1}{6n^{3/2}} - \frac{7}{24n^2} < \left(1+\frac{1}{2n}-\frac{1}{8n^2}\right)\left(\frac{1}{\sqrt{n}} - \frac{1}{3n^{3/2}}\right) < \left(1+\frac{1}{n}\right)^{1/2}\tan^{-1}\frac{1}{\sqrt{n}} < \left(1+\frac{1}{2n}\right)\left(\frac{1}{\sqrt{n}} - \frac{1}{3n^{3/2}} + \frac{1}{5n^{5/2}}\right) < \frac{1}{\sqrt{n}} + \frac{1}{6n^{3/2}} + \frac{1}{25n^2}, \qquad (n \geq 3),$$

and

$$\frac{1}{\sqrt{n}} - \frac{1}{2n^{3/2}} < \frac{1}{\sqrt{n}}\left(1+\frac{1}{n}\right)^{-1/2} < \frac{1}{\sqrt{n}}\left(1-\frac{1}{2n}+\frac{3}{8n^2}\right)$$
$$< \frac{1}{\sqrt{n}} - \frac{1}{2n^{3/2}} + \frac{3}{8n^2}, \qquad (n \geq 3).$$

Substituting the above results in (2.4), we derive the following lemmas.

LEMMA 2.1. *If $n \geq 3$, the following inequality holds:*

$$\pi - \left(\frac{3}{2\sqrt{n}} - \frac{\pi}{2n} + \frac{7}{12n^{3/2}} + \frac{0.4727}{n^2}\right) < \omega(n) < \pi - \left(\frac{35}{24\sqrt{n}} - \frac{\pi}{2n} + \frac{25}{48n^{3/2}} - \frac{0.7865}{n^2}\right), \tag{2.5}$$

where $\omega(n)$ is defined by (1.3).

LEMMA 2.2. *For $n \geq 3$, the following inequalities hold:*

$$\left(\frac{35}{24\sqrt{n}} - \frac{\pi}{2n} + \frac{25}{48n^{3/2}} - \frac{0.7865}{n^2}\right) > \frac{7}{5\left(\sqrt{n}+3\right)}, \tag{2.6}$$
$$\left(\frac{3}{2\sqrt{n}} - \frac{\pi}{2n} + \frac{7}{12n^{3/2}} + \frac{0.4727}{n^2}\right) < \frac{3}{2\left(\sqrt{n}+1\right)}. \tag{2.7}$$

PROOF. For $n \geq 3$, we have

$$\left(\frac{35}{24\sqrt{n}} - \frac{\pi}{2n} + \frac{25}{48n^{3/2}} - \frac{0.7865}{n^2}\right)\left(n^{1/2}+3\right)$$
$$= \frac{35}{24} + \left(\frac{35}{8} - \frac{\pi}{2}\right)\frac{1}{\sqrt{n}} - \left(\frac{3\pi}{2} - \frac{25}{48}\right)\frac{1}{n} + \left(\frac{25}{16} - 0.7865\right)\frac{1}{n^{3/2}} - 2.3595\frac{1}{n^2}$$
$$> \frac{35}{24} + \frac{1}{n}\left(2.5\sqrt{n} - 4.192\right) + \frac{1}{n^2}\left(0.3n^{3/2} + 0.776\sqrt{n} - 2.3595\right) > \frac{35}{24} > \frac{7}{5},$$

so that (2.6) follows.

Similarly, for $n \geq 3$, we obtain

$$\left(\frac{3}{2\sqrt{n}} - \frac{\pi}{2n} + \frac{7}{12n^{3/n}} + \frac{0.4727}{n^2}\right)\left(\sqrt{n}+1\right)$$
$$< \frac{3}{2} - \frac{1}{n^2}\left(0.07n^{3/2} + 0.3n - 0.4727\right) - \frac{1}{n^{3/2}}\left(0.687\sqrt{n} - 1.057\right) < \frac{3}{2},$$

so that (2.7) follows. The lemma is proved.

3. MAIN RESULTS

THEOREM 3.1. *For $n \in N_0$, we have the following inequalities:*

$$\pi - \frac{3}{2\left(\sqrt{n}+1\right)} < \omega(n) < \pi - \frac{7}{5\left(\sqrt{n}+3\right)}, \tag{3.1}$$
$$\frac{7}{5\left(\sqrt{n}+3\right)} < \pi - \sum_{m=0}^{\infty} \frac{1}{(m+n+1)}\sqrt{\frac{n+1}{m+1}} < \frac{3}{2\left(\sqrt{n}+1\right)}. \tag{3.2}$$

PROOF. In view of the results (2.5)–(2.7) and $n \geq 3$, the inequality (3.1) follows.

Since (2.2) holds, we obtain

$$2.5 = \int_0^\infty \frac{1}{(x+1)^{3/2}}\,dx + \frac{1}{2} < \sum_{m=0}^{\infty} \frac{1}{(m+1)^{3/2}} < \int_0^\infty \frac{1}{(x+1)^{3/2}}\,dx + \frac{1}{2} + \frac{1}{8} = 2.625.$$

Then we have

$$\pi - \frac{3}{2(0+1)} < 2.5 < \omega(0) < 2.625 < \pi - \frac{7}{5(0+3)}.$$

By (2.3), we find

$$\begin{aligned}
2.574 < \sqrt{2}\left(\pi - 2\tan^{-1}1 + \frac{1}{4}\right) &< \sum_{m=0}^{\infty} \frac{\sqrt{2}}{(m+2)(m+1)^{1/2}} \\
&< \sqrt{2}\left(\pi - 2\tan^{-1}1 + \frac{13}{24\times 2} + \frac{1}{12\times 4}\right) \\
&< 2.634, \\
2.628 < \sqrt{3}\left(\frac{\pi}{\sqrt{2}} - \frac{2}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{1}{6}\right) &< \sum_{m=0}^{\infty} \frac{\sqrt{3}}{(m+3)(m+1)^{1/2}} \\
&< \sqrt{3}\left(\frac{\pi}{\sqrt{2}} - \frac{2}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{13}{24\times 3} + \frac{1}{12\times 9}\right) < 2.669.
\end{aligned}$$

Thus, we have

$$\pi - \frac{3}{2(1+1)} < 2.574 < \omega(1) < 2.634 < \pi - \frac{7}{5(1+3)}$$

and

$$\pi - \frac{3}{2\left(\sqrt{2}+1\right)} < 2.628 < \omega(2) < 2.669 < \pi - \frac{7}{5\left(\sqrt{2}+3\right)},$$

so that (3.1) holds for any $n \in N_0$. Similarly, (3.2) holds. The theorem is proved.

THEOREM 3.2. *Let $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$, and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$. Then the inequality (1.4) holds and we obtain*

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} \frac{a_n}{m+n+1}\right)^2 < \pi \sum_{n=0}^{\infty}\left[\pi - \frac{7}{5\left(\sqrt{n}+3\right)}\right] a_n^2. \tag{3.3}$$

PROOF. By Cauchy's inequality, we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(m+n+1)} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(m+n+1)^{1/2}}\left(\frac{m+1}{n+1}\right)^{1/4} a_m \frac{1}{(m+n+1)^{1/2}}\left(\frac{n+1}{m+1}\right)^{1/4} b_n \\
&\le \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{m+1}{n+1}\right)^{1/2} a_m^2 \right. \\
&\quad \left. \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2} b_n^2 \right\}^{1/2}.
\end{aligned}$$

By inequality (3.1), (1.4) is valid.

It is obvious that for any $m \in N_0$, $\omega(m) < \pi$. By Cauchy's inequality, we obtain

$$\begin{aligned}
\left(\sum_{m=0}^{\infty} \frac{a_n}{m+n+1}\right)^2 &= \left[\sum_{m=0}^{\infty} \frac{1}{(m+n+1)^{1/2}}\left(\frac{n+1}{m+1}\right)^{1/4} a_n \frac{1}{(m+n+1)^{1/2}}\left(\frac{m+1}{n+1}\right)^{1/4}\right]^2 \\
&\le \sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2} a_n^2 \sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{m+1}{n+1}\right)^{1/2} \\
&= \sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2} a_n^2\, \omega(m) \\
&< \pi \sum_{n=0}^{\infty} \frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2} a_n^2.
\end{aligned}$$

Hence, by (3.1) we find

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^2 &< \pi\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2}a_n^2\\ &= \pi\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{1}{(m+n+1)}\left(\frac{n+1}{m+1}\right)^{1/2}\right]a_n^2\\ &= \pi\sum_{n=0}^{\infty}\omega(n)a_n^2 < \pi\sum_{n=0}^{\infty}\left[\pi-\frac{7}{5\left(\sqrt{n}+3\right)}\right]a_n^2,\end{aligned}$$

so that (3.3) is proved.

REMARK. Inequality (3.2) involves the constant π. Inequalities (1.4) and (3.3) reduce, respectively, to

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\right)^{1/2} \tag{3.4a}$$

and

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{a_n}{m+n+1}\right)^2 < \pi^2\sum_{n=0}^{\infty}a_n^2. \tag{3.4b}$$

These are Hilbert's inequality and its new equivalent form (see [6]). Both (1.4) and (3.3) are strengthened versions of (3.4a) and (3.4b).

REFERENCES

1. S.G. Mikhlin, *Constants in Some Inequalities of Analysis*, John Wiley & Sons, New York, (1986).
2. M. Gao, A note on the Hardy-Hilbert inequality, *J. Math. Anal. Appl.* **204**, 346–351 (1996).
3. B. Yang and M. Gao, On a best value of Hardy-Hilbert's inequality, *Advances in Math.* (China) **26** (2), 159–164 (1997).
4. B. Yang and L. Debnath, Some inequalities involving the constant e, and an application to Carleman's inequality, *J. Math. Anal. Appl.* **222**, 347–353 (1998).
5. K. Knopp, *Theory and Application of Infinite Series*, Blackie & Son Limited, London, (1928).
6. G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, (1952).